

Rudin-Kisler ordering on the P-hierarchy

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Abstract

In [15] M. E. Rudin proved (under CH) that for each P-point u there is a P-point v such that $v >_{RK} u$. In [1] A. Blass improved that theorem assuming MA^1 in the place of CH, in that paper he also proved that under MA^1 each RK-increasing sequence of P-points is upper bounded by a P-point. We improve Blass results simultaneously in 3 directions - we prove it for each class of index ≥ 2 of P-hierarchy (P-points coincidence with a class \mathcal{P}_2 of P-hierarchy), assuming $\mathfrak{b} = \mathfrak{c}$ in the place of MA and we show that there are at least \mathfrak{b} many Rudin-Kisler incomparable such upper bounds.

1 Introduction

We proved in [17] that a class of P-points is precisely a class \mathcal{P}_2 of P-hierarchy which is a classification of ultrafilters on ω into ω_1 disjoint classes. It is natural to ask which properties of the class of P-points are (or are not) also properties of other classes of P-hierarchy. We have started this work in earlier papers [17] and [18] where also Rudin-Kisler ordering was examined. Here, inspired by papers of M. E. Rudin and A. Blass we continue our investigation. The P-hierarchy is defined by monotone sequential contours, and since this ideas are not widely known here we recall all necessary informations.

In [5] S. Dolecki and F. Mynard introduced monotone sequential cascades - special kind of trees - as a tool to describe topological sequential spaces. Cascades and their contours appeared to be also an useful tool to investigate certain types of ultrafilters on ω , namely ordinal ultrafilters and the P-hierarchy (see [17], [18]).

1) The theorem was stated under MA, but in fact the Blass proof works also under $\mathfrak{p} = \mathfrak{c}$, which was mentioned in [1]

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The *cascade* is a tree V , ordered by " \sqsubseteq ", without infinite branches and with the minimal element \emptyset_V . A cascade is *sequential* if for each non-maximal element of V ($v \in V \setminus \max V$) the set v^{+V} of immediate successors of v (in V) is countably infinite. We write v^+ instead of v^{+W} if it is known in which cascade the successors of v are considered. If $v \in V \setminus \max V$, then the set v^+ (if infinite) may be endowed with an order of the type ω , and then by $(v_n)_{n \in \omega}$ we denote the sequence of elements of v^+ , and by $v^{(n)W}$ - the n -th element of v^{+W} .

The *rank* of $v \in V$ ($r_V(v)$ or $r(v)$) is defined inductively as follows: $r(v) = 0$ if $v \in \max V$, and otherwise $r(v)$ is the least ordinal greater than the ranks of all immediate successors of v . The rank $r(V)$ of the cascade V is, by definition, the rank of \emptyset_V . If it is possible to order all sets v^+ (for $v \in V \setminus \max V$) so that for each $v \in V \setminus \max V$ the sequence $(r(v^{(n)}))_{n < \omega}$ is non-decreasing (in other words if for each $v \in V \setminus \emptyset_V$ the set $\{v \in (w)^+ : r(v) < \alpha\}$ is finite for each $\alpha < r(w)$), then the cascade V is *monotone*, and we fix such an order on V without indication.

For $v \in V$ by v^\uparrow we understand $\{w \in V : v \sqsubseteq w\}$ with preserved order, if V is a monotone sequential cascade and $U \# \int V$ then by $V^{\downarrow V}$ we understand the biggest monotone sub-cascade of cascade V such that for each element $w \in V^{\downarrow V}$ we have $\max(w^\uparrow) \in U$.

Let W be a cascade, and let $\{V_w : w \in \max W\}$ be a set of pairwise disjoint cascades such that $V_w \cap W = \emptyset$ for all $w \in \max W$. Then, the *confluence* of cascades V_w with respect to the cascade W (we write $W \leftarrow P V_w$) is defined as a cascade constructed by an identification of $w \in \max W$ with \emptyset_{V_w} and according to the following rules: $\emptyset_W = \emptyset_{W \leftarrow P V_w}$; if $w \in W \setminus \max W$, then $w^{+W \leftarrow P V_w} = w^{+W}$; if $w \in V_{w_0}$ (for a certain $w_0 \in \max W$), then $w^{+W \leftarrow P V_w} = w^{+V_{w_0}}$; in each case we also assume that the order on the set of successors remains unchanged. By $(n) \leftarrow P V_n$ we denote $W \leftarrow P V_w$ if W is a sequential cascade of rank 1.

If $\mathbb{U} = \{u_s : s \in S\}$ is a family of filters on X and if p is a filter on S , then the *contour of $\{u_s\}$ along p* is defined by

$$\int_p \mathbb{U} = \int_p u_s = \bigcup_{P \in p} \bigcap_{s \in P} u_s.$$

Such a construction has been used by many authors ([7], [8], [9]) and is also known as a sum (or as a limit) of filters.

For the sequential cascade V we define the *contour* of V (we write $\int V$) inductively: if $r(V) = 1$ then $\int V$ is a co-finite filter on $\max(V)$, if $W = V \leftarrow P V_w$ then $\int W = \int_V \int V_w$. Similar filters were considered in [10], [11], [3]. Let V be a monotone sequential cascade and let $u = \int V$. Then the

$\text{rank } r(\int V)$ of $\int V$ is, by definition, the rank of V . It was shown in [6], that for each countable ordinal $\alpha \geq 1$, there is a monotone sequential contour of rank α . It was shown in [6] that if $\int V = \int W$, then $r(V) = r(W)$. The reader may find more information about monotone sequential cascades and their contours in [4], [5], [6], [16], [18], [17].

We say that an ultrafilter u belongs to a class \mathcal{P}_α (we write $u \in \mathcal{P}_\alpha$) if

1) for each $\beta < \alpha$ there is a monotone sequential contour of rank β contained in u

2) there is no monotone sequential contour of rank α contained in u .

Although this paper is self-contained we suggest to look at [18], [17] for more information concerning P-hierarchy.

IMPORTANT: In the remainder of this paper each filter is considered to be on ω , unless indicated otherwise, and for $f, g : \omega \rightarrow \omega$ we say that f *dominates* g if $f(n) > g(n)$ for all(!) $n < \omega$; this understanding does not change a domination number \mathfrak{d} .

2 Results

Let V be a monotone sequential cascade of rank ≥ 2 . If from V we remove all branches of height 1 obtaining a cascade W then $\int V = \int W$. Thus we assume that each cascade of rank ≥ 2 has no branches of height 1.

Let V be a sequential cascade, we classically identify elements of the cascade with finite sequence of naturals by a function $f : V \rightarrow \omega^{<\omega}$ as follows:

$f(\emptyset_V) = \emptyset$; $f(w) = f(v) \frown n$ if w is the n -th element of v^+ . As a convention, we identify v with $f(v)$, and see the cascade as a subset of $\omega^{<\omega}$.

A sequential cascade V is *absorbing* if it fulfills the following condition: if (a_1, \dots, a_n) belongs to V and $b_i \geq a_i$ for each $i \in \{1, \dots, n\}$ then $(b_1, \dots, b_n) \in V$. Note that each absorbing cascade is monotone. A contour of the absorbing cascade is called an absorbing contour.

Remark 2.1. *For each monotone sequential cascade V of rank less than, or equal to ω there is an absorbing cascade W such that $\int V = \int W$. (It was proved in [6] that then $r(V) = r(W)$).*

Proof. For V of finite rank, it suffices to remove all branches of height less than $r(V)$ and re-enumerate branches. Take a monotone sequential cascade V of rank ω , let $V = (n) \dot{\cup} V_n$ and in each V_n remove all branches of height less than $r(V_n)$ to obtain the cascade we looking for (after re-enumerating of branches). ■

We do not know weather we can extended Remark 2.1 to cascades of higher ranks ¹, but we have a little weaker Theorem 2.5 for them; first we need a lemma where by $-1 + \gamma$ we denote γ if γ is infinite, $\gamma - 1$ if $\gamma < \omega$.

Lemma 2.2. *For each countable ordinal γ there is a $(-1 + \gamma + 1)$ -sequence $((a_{\alpha,\gamma}^n)_{n<\omega})_{1<\alpha\leq\gamma}$ of non decreasing ω -sequences of ordinal numbers, such that $\lim_{n<\omega}(a_{\alpha,\gamma}^n + 1) = \alpha$ and $\alpha < \beta < \gamma$ implies $a_{\alpha,\gamma}^n \leq a_{\beta,\gamma}^n$ for each natural number n .*

Proof. In the contrary. Let γ be the first ordinal, such that the claimed sequence does not exist.

If $\gamma = \delta + 1$ for some δ , then for each n it suffices to take : $a_{\alpha,\gamma}^n = a_{\alpha,\delta}^n$ for each $\alpha \leq \delta$, and $a_{\gamma,\gamma}^n = \delta$.

If γ is a limit, take an increasing sequence (γ_n) of ordinals, such that $\gamma_1 = 1$ and $\lim_{n<\omega}(\gamma_n + 1) = \gamma$. We define:

$$a_{\alpha,\gamma}^2 = 1$$

for α such that $\gamma_n < \alpha \leq \gamma_{n+1}$, for $m \leq n$ let $a_{\alpha,\gamma}^m = \gamma_m$

for α such that $\gamma_n < \alpha \leq \gamma_{n+1}$, for $m > n$ let $a_{\alpha,\gamma}^m = \max\{\gamma_n, a_{\alpha,\gamma_{n+1}}^m\}$

Standard check shows that the defined sequence fulfills the claim. ■

In forthcoming paper [18] we showed the following Remark 2.3 with standard proof by induction with respect to rank.

Remark 2.3. [18] *For each monotone sequential cascade V , for each ordinal $\alpha < r(V)$ there exists a monotone sequential cascade W of rank α such that $\int W \subset \int V$.*

Lemma 2.4 (Folklore). *Let (a_n) and (b_n) be nondecreasing sequences of ordinals such that $a_1 = b_1$ and $\lim_{n<\omega} a_n = \lim_{n<\omega} b_n$. Then there is a non-decreasing, finite-to-one suriection $f : \omega \rightarrow \omega$ such that $b_n \geq a_{f(n)}$.*

Proof. : Put $f(1) = 1$, now let $f(n) = f(n-1) + 1$ if $a_{f(n-1)+1} \leq b_n$, and $f(n) = f(n-1)$ otherwise. ■

Theorem 2.5. *For each monotone sequential cascade V there is a absorbing cascade W such that $r(V) = r(W)$ and $\int W \subset \int V$.*

Proof. Fix γ and a $(-1 + \gamma + 1)$ -sequence $((a_{\alpha,\gamma}^n)_{n<\omega})_{1<\alpha\leq\gamma}$ from the Lemma 2.2. We will show a little more, i.e. that for each monotone sequential cascade V of rank γ there is a absorbing cascade W such that: $r(w_n^+) = a_{r(w),\gamma}^n$ for each $w \in W \setminus \max(W)$.

Since a set $\{v : v \in \emptyset_V^+, r(v) < a_{r(V),\gamma}^1\}$ is finite, so without loss of generality, we can assume that $r(V) \geq a_{r(V),\gamma}^1$ for all $v \in \emptyset_V^+$.

1) We suppose that there is a counter-example for each $\alpha > \omega$

Put $b_n = r(\emptyset_V^{+,n})$ and $a_n = a_{\gamma,\gamma}^n$ and fix a function f from the Lemma 2.4.

By Remark 2.3 for each $n < \omega$ there is a monotone sequential cascade T_n such that $r(T_n) = a_{\gamma,\gamma}^n$ and $\int T_n \subset \int V_n$.

Let K_n be a cascade obtained from cascades T_m for $m \in f^{-1}(n)$ by identifying all such \emptyset_{T_m} , i.e. $\emptyset_{K_n}^+ = \bigcup_{m \in f^{-1}(n)} \emptyset_{T_m}^+$, and if $k \in K$, $k \neq \emptyset_K$ then $k \in T_m$ for some $m \in f^{-1}(n)$ and $k^{+k} = k^{+T_m}$. Now, for each K_n we use an inductive assumption, obtaining W_n , and the confluence $(k) \leftarrow \wp K_k$ is the cascade we are looking for. ■

Corollary 2.6. *An ultrafilter u contains a contour of a monotone sequential cascade of rank α if and only if u contains a contour of an absorbing cascade of the rank α .*

Let V be a sequential cascade, and let $f, g : V \setminus \max(V) \rightarrow \omega$. We say that a function f *V-dominates* g (in symbols $f \geq_V g$) if there is $U \in \int V$ such that $f(v) \geq g(v)$ for each $v \in U^\downarrow$. In this meaning we define the V -dominating family and analogically V -dominating number \mathfrak{d}_V . Define also the set $V(f)$ inductively: $\emptyset_V \in V(f)$ and if $v \in V(f)$ then $v \smallfrown k \in V(f)$ if $k \geq f(v)$. For $U \in \int V$ we define in the analogical way $f_U : \text{dom}(f_U) \rightarrow \omega$: $\emptyset_V \in \text{dom}(f_U)$, if $v \in \text{dom}(f_U)$ then $f(v) = \min \{n < \omega : \forall m \geq n, \int (v \smallfrown n)^\uparrow \# U\}$; $v \smallfrown m \in \text{dom}(f_U)$ for all $m \geq f_U(v)$.

Remark 2.7. *For an absorbing cascade V a family \mathbb{F} of functions $V \setminus \max V \rightarrow \omega$ is V -dominating if and only if a family $\{V(f) : f \in \mathbb{F}\}$ is a base of $\int V$.*

For an absorbing cascade V and for a function $f : V \setminus \max V \rightarrow \omega$, we define inductively a partial function $f_{\text{Shift}} : V \supset \text{dom}(f_{\text{Shift}}) \rightarrow V$ as follows:

$\emptyset_V \in \text{dom}(f_{\text{Shift}})$ and $f_{\text{Shift}}(\emptyset_V) = \emptyset_V$;

if $v \in \text{dom}(f_{\text{Shift}})$ and $f_{\text{Shift}}(v) \notin \max V$ then $(v \smallfrown k) \in \text{dom}(f_{\text{Shift}})$ for $k \geq f(v)$ and $f_{\text{Shift}}(v \smallfrown k) = f_{\text{Shift}}(v) \smallfrown (k - f(v) + 1)$.

Note that f_{Shift} is a bijection and $\text{rng}(f_{\text{Shift}}) = V$.

If g is also a function $V \setminus \max(V) \rightarrow \omega$, then $f_{\text{Shift}}(g)(v) : V \rightarrow \omega$ is defined by: $f_{\text{Shift}}(g)(v) = g(f_{\text{Shift}})^{-1}(v)$.

Theorem 2.8. *Let V be an absorbing cascade, a family $\mathbb{F} \subset \omega^{V \setminus \max(V)}$ is V -dominating if and only if $\mathbb{F}^* = \{f_{\text{Shift}}(f) : f \in \mathbb{F}\}$ is a dominating family on $V \setminus \max(V)$.*

Proof. Let $\mathbb{F} \subset \omega^{V \setminus \max(V)}$ be a V -dominating family, and let $f : V \setminus \max(V) \rightarrow \omega$. Take $g : V \setminus \max(V) \rightarrow \omega$ defined as follows: $g(a_1, \dots, a_n) = \max \{f((b_1, \dots, b_n)); b_i \leq a_i, (b_1, \dots, b_n) \in V\}$. Since \mathbb{F} is V -dominating, there is $h \in \mathbb{F}$ that V -dominates g . Clearly $h_{\text{Shift}}(h) \geq f$.

Now let f be a witness that \mathbb{F}^* is not dominating on V , define g as above and observe that g is not V -dominating by any $f \in \mathbb{F}$. ■

By Corollary 2.6 Remark 2.7 and Theorem 2.8 we have:

Corollary 2.9. *For each absorbing cascade V there is $\mathfrak{d}_V = \mathfrak{d}$.*

Remark 2.10 (Folklore). *The minimum of cardinalities of families of non-dominating families such that the sum of all that families is a dominating family is \mathfrak{b}*

Proof. For each α , $\alpha < \lambda < \mathfrak{b}$, let \mathbb{F}_α be a non-dominating family of functions $\omega \rightarrow \omega$. Let f_α be a function non dominated by \mathbb{F}_α , let f be a function that dominates all f_α (there is some since $\lambda < \mathfrak{b}$). Clearly f can not be dominated by any element of $\bigcup_{\alpha < \lambda} \mathbb{F}$.

Let $(f_\alpha)_{\alpha < \mathfrak{b}}$ be a non-limited sequence of functions. Define \mathbb{F}_α as a family of all functions f such that f does not dominate $\{f_\beta : \beta \leq \alpha\}$. Clearly \mathbb{F}_α is non-dominating, and $\bigcup_{\alpha < \mathfrak{b}} \mathbb{F}_\alpha = \omega^\omega$ and so is dominating. ■

Let $\mathbb{A} \subset {}^X 2$, a *supersets closure* of \mathbb{A} is a family $\text{SC}(\mathbb{A}) = \bigcup_{A \in \mathbb{A}} \langle A \rangle$. Let u be a filter on X , we say that a family $\mathbb{P} \subset 2^X$ is a π -base of u if \mathbb{P} has finite intersection property, and if $u \subset \text{SC}(\mathbb{P})$.

Corollary 2.11. *A sum of less than \mathfrak{b} families \mathbb{A}_α that do not contain a π -base of absorbing contour of rank $\alpha > 1$ does not contain any π -base of monotone sequential contour of rank α*

Proof. Suppose on the contrary that there is a pair of witnesses - a sequence $(\mathbb{A}_\alpha)_{\alpha < \lambda < \mathfrak{b}}$ and a monotone sequential cascade W , and fix a classical - defined by functions $V \setminus \max(V) \rightarrow \omega$ - base \mathbb{B} of V . By Theorem 2.5 or by Remark 2.1 (depending on the rank of the cascade) there is an absorbing cascade V of the rank $r(V) = r(W)$ such that $\int V \subset \int W$, clearly each π -base of $\int W$ is also a π -base of V , so it suffices to prove this corollary for absorbing cascades. Since a family \mathbb{A}_α does not contain a π -base of $\int V$ and since \mathbb{A}_α is a π -base of $\mathbb{B}_\alpha = \{B \in \mathbb{B} : B \supset A, A \in \mathbb{A}_\alpha\}$, thus \mathbb{B}_α does not contain $\int V$, and therefore by Remark 2.7 a family $\{f_B; B \in \mathbb{B}_\alpha\}$ is not V -dominating for each $\alpha < \lambda$. By Theorem 2.8 $\{f_{B\text{Shift}}(f_B); B \in \mathbb{B}_\alpha\}$ is not dominating so by Remark 2.10 a family $\bigcup_{\alpha < \lambda} \{f_{B\text{Shift}}(f_B); B \in \mathbb{A}_\alpha\}$ is not dominating, and by Theorem 2.8 $\bigcup_{\alpha < \lambda} \{f_B; B \in \mathbb{B}_\alpha\}$ is not V -dominating. Therefore by Remark 2.7 a family $\bigcup_{\alpha < \lambda} \mathbb{B}_\alpha$ does not contain a base of $\int V$, but since $\bigcup_{\alpha < \lambda} \mathbb{B}_\alpha$ contains all supersets of elements of $\bigcup_{\alpha < \lambda} \mathbb{A}_\alpha$ which belong to \mathbb{B} thus $\bigcup_{\alpha < \lambda} \mathbb{B}_\alpha$ does not contain a π -base of $\int V$, and since $\bigcup_{\alpha < \lambda} \mathbb{A}_\alpha \subset \bigcup_{\alpha < \lambda} \mathbb{B}_\alpha$ thus $\bigcup_{\alpha < \lambda} \mathbb{A}_\alpha$ does not contain a π -base of $\int V$. ■

Corollary 2.12. *An increasing (\subset) sequence of length less than \mathfrak{b} of filters that do not contain a π -base of an absorbing sequential contour of rank $\alpha > 1$ does not contain any monotone sequential contour of rank α .*

Remark 2.13. *If v is a filter and T is a set such that $T \# v$ then if $v \restriction_T$ does not contain any monotone sequential contour of rank α then v does not contain any π -base of any monotone sequential contour of rank α .*

Remark 2.14. *Let u be a filter which does not contain any π -base of any monotone sequential cascade of rank α , and let $f : \omega \rightarrow \omega$, then a filter $\langle \{f^{-1}[U] : U \in u\} \rangle$ does not contain any π -base of any monotone sequential cascade of rank α .*

Proof. For each n such that $f^{-1}(n)$ is nonempty put $x_n = \min(f^{-1}(n))$ and let X be a set of all such x_n 's. It suffices to consider $\langle \{f[U] : U \in u\} \rangle \restriction_X$ which is a copy of u . ■

Theorem 2.15. *[17, Theorem 2.5] Let $(\alpha_n)_{n < \omega}$ be a non-decreasing sequence of ordinals less than ω_1 , let $\alpha = \lim_{n < \omega} (\alpha_n)$, let $1 < \beta < \omega_1$. If $u_n \in \mathcal{P}_{\alpha_n}$ is a discrete sequence of ultrafilters and $u \in \mathcal{P}_\beta$ then $\int_u u_n \in \mathcal{P}_{\alpha + (-1 + \beta)}$.*

Remark 2.16. *Let u be such a filter that there is a map $\omega \rightarrow \omega$ that $f(u) = \mathfrak{fr}$. If $\langle u \cup \mathbb{A} \rangle$ is an ultrafilter for some family (of sets) \mathbb{A} then $\text{card}(\mathbb{A}) \geq \mathfrak{u}$.*

Proof. Let $f : \omega \rightarrow \omega$ be a function, such that $f(u) = \mathfrak{fr}$, if $\langle u \cup \mathbb{A} \rangle$ is an ultrafilter thus $f(\langle u \cup \mathbb{A} \rangle)$ is a free ultrafilter and so $\{f[A] : A \in \mathbb{A}\}$ is a base of ultrafilter i.e. it has a cardinality of at least \mathfrak{u} . ■

Theorem 2.17. *($\mathfrak{b} = \mathfrak{c}$) Let $1 < \xi \leq \omega_1$ and let $p \in \mathcal{P}_\xi$, then there is $\mathfrak{U} \subset \mathcal{P}_\alpha$ of cardinality \mathfrak{b} that $u >_{RK} p$ for each $u \in \mathfrak{U}$, and that elements of \mathfrak{U} are Rudin-Kisler incomparable.¹⁾*

Proof.

For $\xi = \omega_1$ the claim is obvious. Fix $1 < \xi < \omega_1$. Let $f : \omega \rightarrow \omega$ be a finite-to-one function such that $\sup\{n : n \in U\} = \omega$ for each $U \in p$.

Let \mathcal{A}_n be a family of such subsets of ω that there is $P \in p$ that $\text{card}(f^{-1}(m)) = \text{card}(f^{-1}(m) \cap A) + n$ for each $m \in P$.

1) We can obtain an easier version of the Theorem in a much shorter way: (P-points exists) Let α be infinite countable ordinal, than for each $u \in \mathcal{P}_\alpha$ there is $v \in \mathcal{P}_\alpha$ such that $v >_{RK} u$.

Proof. Let $\alpha \geq \omega$, and take any $u \in \mathcal{P}_\alpha$. Consider a partition (A_n) of ω into ω infinite sets. For each n let u_n be a P-point such that $A_n \in u_n$. Put $v = \int_u u_n$, by Theorem 2.15 $v \in \mathcal{P}_\alpha$, on the other hand for a function $f(m) = n$ for $m \in A_n$ we have $f(v) = u$ and there is no set $V \in v$ that $f \restriction_V$ is one-to-one, so it can not be Rudin-Kisler equivalent.

For $\alpha = \omega_1$ the claim is obvious. ■

Proposition 2.18. *For each natural number i a family $\mathbb{B}_i = \{f^{-1}[P] \cap A^c : P \in p, A \in \mathcal{A}_i\}$ does not contain a π -base of any monotone sequential contour of rank ξ .*

On the contrary. Let $i = \min\{j < \omega : \text{there is a } \pi\text{-base monotone sequential cascade of rank } \alpha \text{ contained in } \mathbb{B}_j\}$, let \mathbb{P} be a π -base of absorbing sequential contour $\int V$ of rank ξ , such that $\mathbb{P} \subset \mathbb{B}_j$. For each $U \in \int V$, for each $k \leq i$ let $W_k(U) = \{n \in \omega : \text{card}(f^{-1}(m)) = \text{card}(f^{-1}(m) \cap U) + k\}$, since p is an ultrafilter, then for each $U \in \int V$ there is $k(U) \leq i$ such that $W_{k(U)}(U) \in p$. Define also $A_k(U) = \{n < \omega : n \in f^{-1}(m) \text{ for such } m \text{ that } \text{card}(f^{-1}(m)) = \text{card}(f^{-1}(m) \cap U) + k \text{ and } n \notin U\}$. By minimality of i there is $U_0 \in V$ such that $k(U_0) = i$, moreover if $U_1 \subset U_0$ then $k(U_1) = i$ and $A_i(U_1) \subset A_i(U_0)$. Thus $\int V \subset \text{SC}(\{f^{-1}[P] \cap A_i(U_0)^c : P \in p\})$ what is impossible since $\text{SC}(\{f^{-1}[P] \cap A_i(U_0)^c : P \in p\}) = \text{SC}(\{(f|_{A_i(U_0)^c})^{-1}[P] : P \in p\})$ so by Remark 2.14 it does not contain any monotone sequential cascade of rank ξ . ■

Proposition 2.19. *$\langle f^{-1}[P] \cap A^c : P \in P, A \in \mathcal{A} \rangle$ do not contain any monotone sequential contour of rank ξ .*

By Proposition 2.18 each $\mathbb{B}_i = \{f^{-1}[P] \cap A^c : P \in p, A \in \mathcal{A}_i\}$ do not contain a π -base of any monotone sequential contour of rank ξ , thus by Corollary 2.11 $\bigcup_{n < \omega} \mathbb{B}_i$ does not contain a π -base of any monotone sequential contour of rank ξ , but $\bigcup_{n < \omega} \mathbb{B}_i = \langle f^{-1}[P] \cap A^c : P \in P, A \in \mathcal{A} \rangle$, i.e., is a filter and so since it does not contain a π -base of any monotone sequential contour of rank ξ , it also does not contain any monotone sequential contour of rank ξ . ■

Clearly there is no set \bar{A} that $\langle f^{-1}[P] \cap A^c \cap \bar{A} : P \in P, A \in \mathcal{A} \rangle$ is an ultrafilter thus there is a sequence of pairwise disjoint sets $(C_n)_{n < \omega}$ such that $C_n \# \langle f^{-1}[P] \cap A^c : P \in P, A \in \mathcal{A} \rangle$ and $\bigcup_{n < \omega} C_n = \omega$. Let $\mathcal{T} = \langle \{f^{-1}[P] \cap A^c : P \in P, A \in \mathcal{A}\} \cup \{\bigcup_{n > m} C_n : m < \omega\} \rangle$.

Remark 2.20. *A filter $\langle \mathcal{T} \cup \bar{A} \rangle$ dose not contain any monotone sequential contour of rank ξ for any $\bar{A} \# \mathcal{T}$, and there is a function $f : \omega \rightarrow \omega$ that $f(\mathcal{T}) = \mathfrak{F}\mathfrak{r}$.*

First part follows from Proposition 2.19, and a number of generators added to the family $\langle f^{-1}[P] \cap A^c \cap \bar{A} : P \in P, A \in \mathcal{A} \rangle$ in the virtue of Corollary 2.11. To see the second part of Remark 2.20 it suffices to consider a function $f : \omega \rightarrow \omega$ such that $f(n) = m$ if $n \in C_m$. ■

We enlist all absorbing cascades of rank ξ in a sequence $(V_\alpha)_{\alpha < \mathfrak{b}}$ and all functions $\omega \rightarrow \omega$ in a sequence $(f_\beta)_{\beta < \mathfrak{b}}$. We will build a family $\{(\mathcal{F}_\alpha)^\beta\}_{\beta < \mathfrak{b}}$ of increasing \mathfrak{b} -sequences $(\mathcal{F}_\alpha)_{\alpha < \mathfrak{b}}$ of filters such that:

- 1) Each \mathcal{F}_α^β is generated by \mathcal{T} together with some family of cardinality $< \mathfrak{b}$ of sets;
- 2) $\mathcal{F}_0^\beta = \mathcal{T}$ for each $\beta < \mathfrak{b}$;
- 3) For each $\alpha, \beta < \mathfrak{b}$, there is $F \in \mathcal{F}_{\alpha+1}^\beta$ such that $F^c \in \int V_\alpha$;
- 4) For limit α for each β , $\mathcal{F}_\alpha^\beta = \bigcup_{\gamma < \alpha} \mathcal{F}_\gamma^\beta$;
- 5) For each α , for each $\gamma < \alpha$, for each $\beta_1, \beta_2 < \alpha$ there is a set $F \in \mathcal{F}_{\alpha+1}^{\beta_1}$, such that $(f_\gamma[F])^c \in \mathcal{F}_{\alpha+1}^{\beta_2}$.

Existence of such families is a standard work by induction with respect to α with sub-induction with respect to $\gamma < \alpha$, with sub-sub induction with respect to $\beta_1 < \gamma$ and with sub-sub-sub-induction with respect to $\beta_2 < \beta_1$, using Remark 2.16, Remark 2.20 and Remark 2.10.

Now it suffice for each $\beta < \mathfrak{b}$ take any ultrafilter extending $\bigcup_{\beta < \mathfrak{b}} \mathcal{F}_\alpha^\beta$. ■

Theorem 2.21. ($\mathfrak{b} = \mathfrak{c}$) *Let $1 < \xi \leq \omega_1$, and let $(p_n)_{n < \omega}$ be a RK-increasing sequence of elements of \mathcal{P}_ξ , then there exists $u \in \mathcal{P}_\xi$ such that $u >_{RK} p_n$ for each $n < \omega$.*

Proof. Let f_n be a function $\omega \rightarrow \omega$ - witness that $p_{n+1} >_{RK} p_n$. For each natural number m consider on $\omega \times \omega$ a family of sets \mathbb{B}_m such that $\mathbb{B}_m \mid (\omega \times \{n\}) = \langle \{f_{n-1}^{-1} \circ f_{n-2}^{-1} \circ \dots \circ f_m^{-1}(P) : P \in p_n\} \rangle$ for $n \geq m$. Let $\mathbb{B} = \bigcup_{m < \omega} \mathbb{B}_m$. Clearly \mathbb{B} is a filter, and each ultrafilter which extends \mathbb{B} is RK greater then each p_n . Also, by Remark 2.14, each \mathbb{B}_n does not contain any monotone sequential contour of rank ξ so by Corollary 2.12, \mathbb{B} does not contain any monotone sequential contour of rank ξ .

We enlist all absorbing cascades of rank ξ in a sequence $(V_\alpha)_{\alpha < \mathfrak{b}}$ and we will build an increasing \mathfrak{b} -sequence of filters \mathcal{F}_α such that:

- 1) $\mathcal{F}_0 = \mathbb{B}$.
- 2) For each α , there is such $F \in \mathcal{F}_{\alpha+1}$ that $F^c \in \int V_\alpha$;
- 3) For a limit α , $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$.

The rest of the proof is an easier version of the final part of the proof of Theorem 2.17. ■

Corollary 2.22. ($\mathfrak{b} = \mathfrak{c}$) *Let $1 < \xi \leq \omega_1$, and let $(p_n)_{n < \omega}$ be a RK-increasing sequence of elements of \mathcal{P}_ξ , then there exists a family $\mathfrak{U} \subset \mathcal{P}_\xi$ of cardinality \mathfrak{b} such that $u >_{RK} p_n$ for each $u \in \mathfrak{U}$, and that elements of \mathfrak{U} are Rudin-Kisler incomparable.*

Proof. Just combine Theorem 2.21 with Theorem 2.17. ■

Theorem 2.23. [17, Theorem 2.8] *The following statements are equivalent:*

- 1) *P-points exist;*
- 2) *Classes \mathcal{P}_α are nonempty for each countable successor α ;*
- 3) *There exists a countable successor $\alpha > 1$ such that the class \mathcal{P}_α is nonempty.*

Theorem 2.24. [12] $\mathfrak{d} = \mathfrak{c}$ if, and only if, every filter generated by less than \mathfrak{c} elements can be extended to a P -point.

Theorem 2.25. ($\mathfrak{b} = \mathfrak{c}$) Each class of \mathcal{P} -hierarchy is non-empty.

Proof. For classes of index $\xi \in \{1, \omega_1\}$ we are done (in ZFC) by [18, Corollary 6.4]

For successor $1 < \xi < \omega_1$, since $\mathfrak{b} \leq \mathfrak{d}$, it suffice to combine Theorem 2.23 and Theorem 2.24.

For limit $\xi < \omega_1$ a proof is essentially the same as final part of the proof of Theorem 2.17:

We enlist all absorbing cascades in a sequence $(V_\alpha)_{\alpha < \mathfrak{b}}$. Let $(\int V_n)$ be an increasing (" \subset ") sequence of monotone sequential contours such that $\lim_{n < \omega} (r(V_n) + 1) = \xi$, such sequences exist in ZFC - and were constructed in the proof of [6, Theorem 4.6]. Thus by Corollary 2.11 $\bigcup_{n < \omega} \int V_n$ does not contain a π -base of monotone sequential cascade of rank ξ .

We enlist all absorbing cascades of rank ξ in a sequence $(V_\alpha)_{\alpha < \mathfrak{b}}$ and we will build an increasing \mathfrak{b} -sequence of filters \mathcal{F}_α such that:

- 1) $\mathcal{F}_0 = \bigcup_{n < \omega} \int V_n$.
- 2) For each α , there is $F \in \mathcal{F}_{\alpha+1}$ such that $F^c \in \int V_\alpha$;
- 3) For a limit α , $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$.

The rest of the proof is an easier version of the final part of the proof of Theorem 2.17. ■

Corollary 2.26. ($\mathfrak{b} = \mathfrak{c}$) Each class of \mathcal{P} -hierarchy of index > 1 contains a family of cardinality \mathfrak{b} of pairwise Rudin-Kisler incomparable ultrafilters.

Proof. Just combine Theorem 2.25 with Theorem 2.17. ■

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